Section 5.2

The Definite Integral

(1) The Definite Integral and Net Area

(2) Properties of the Definite Integral

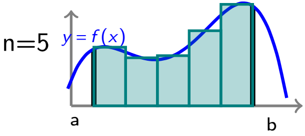
(3) Evaluating Definite Integrals using Limits

The Definite Integral of a Function

The **definite integral** of a function f on the interval $[a, b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

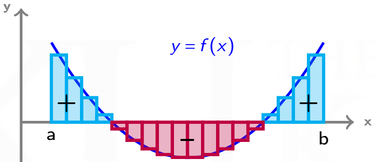
(where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$), provided that this limit exists.



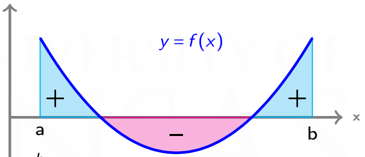
The integral symbol $\int$ is an elongated S, introduced by Leibniz because an integral is a limit of sums.

- The procedure of calculating an integral is **integration**.
- $f(x)$ is the **integrand** and $\{a, b\}$ are the **limits of integration**.
- If $\int_a^b f(x)\,dx$ is defined, we say that f is **integrable** on $[a, b]$.

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is **integrable** on $[a, b]$.

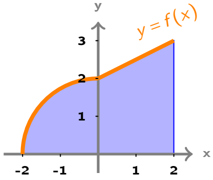


$\sum f(x_i)\Delta x$ is an approximation of the net area.



$\int_a^b f(x)\,dx$ is the net area.

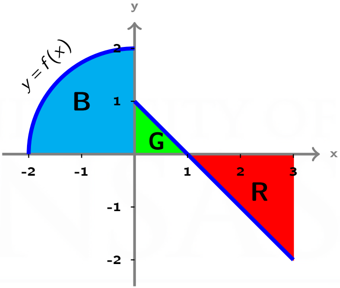
The definite integral calculates **net area:** the area below the positive part of a graph **minus** the area above the negative part.



Example 1: Evaluate $\int_{-2}^{2} f(x)\,dx$ using geometry.

Net Area and Total Area

Example 2: For the function $f(x)$ shown to the right, evaluate $\int_{-2}^{3} f(x)\,dx$.



Properties of the Definite Integral

Integral Property #1

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

For the first integral, the base of each rectangle in the Riemann sum has length $\frac{b-a}{n}$, while in the second integral each rectangle has base $\frac{a-b}{n} = -\frac{b-a}{n}$.

Integral Property #2

$$\int_a^a f(x)\,dx = 0$$

The interval $[a, a]$ has length 0, so the region above or below the graph is just a line segment, which has area 0.

Properties of the Definite Integral

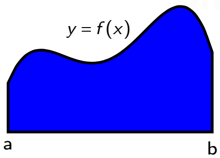
Integral Property #3

$$\int_a^b c\,dx = c(b-a)$$

If $f(x) = c$, then the area below the graph of f is a rectangle with base $b-a$, height c, and therefore area $c(b-a)$.

Integral Property #4

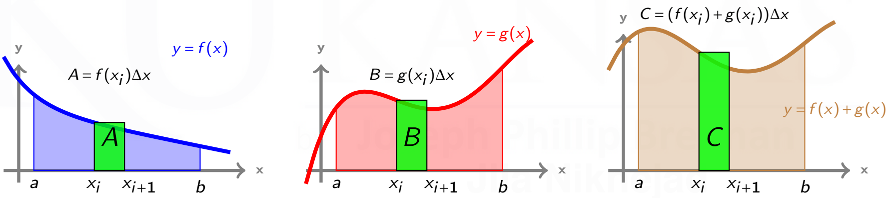
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Properties of the Definite Integral

Integral Property #5

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



The brown area is the sum of the blue and the red area.

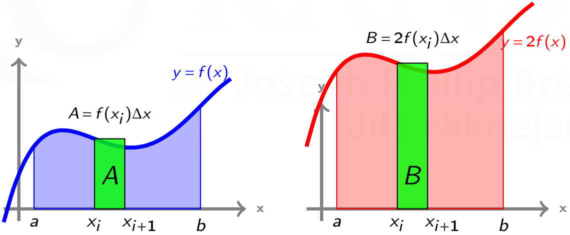
$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Properties of the Definite Integral

Integral Property #6

$$\int_a^b (cf(x))\,dx = c\int_a^b f(x)\,dx \text{ for any constant } c.$$

Stretching a graph vertically by a factor of c multiplies net area by c.



The red area is twice as the blue area.

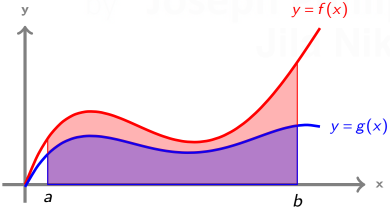
Integral Comparison Property #1

If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

(In this case, net area just means total area under the curve.)

Integral Comparison Property #2

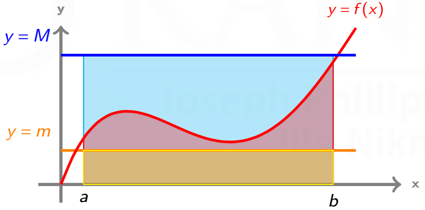
If $f(x) \geq g(x)$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.



Integral Comparison Property #3

If $m \le f(x) \le M$ on the interval $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$



Evaluating Definite Integrals as Limits

The **definite integral** of f on the interval $[a, b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$, provided that this limit exists.

Example 3: Express the definite integral

$$\int_1^5 \frac{2x}{1-x^3}\,dx$$

as a limit of Riemann sums.

Example 4: What definite integral is represented by

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(2 + \frac{3i}{n}\right) \sin\left(2 + \frac{3i}{n}\right) \right) \frac{3}{n} \;?$$

Example 5: Evaluate the integral $\int_1^3 \left(x^2 - 6\right) dx$.